ON COHEN–MACAULAY MODULES OVER NON-COMMUTATIVE SURFACE SINGULARITIES

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Abstract. We generalize the results of Kahn about a correspondence between Cohen–Macaulay modules and vector bundles to non-commutative surface singularities. As an application, we give examples of non-commutative surface singularities which are not Cohen–Macaulay finite, but are Cohen–Macaulay tame.

Contents

Introduction

Cohen–Macaulay modules over commutative Cohen–Macaulay rings have been widely studied. A good survey on this topic is the book of Yoshino [14]. In particular, for curve, surface and hypersurface singularities criteria are known for them to be *Cohen–Macaulay finite*, i.e. only having finitely many indecomposable Cohen–Macaulay modules (up to isomorphism). For curve singularities and minimally elliptic surface singularities criteria are also known for them to be *Cohen–Macaulay tame*, i.e. only having 1-parameter families of non-isomorphic indecomposable Cohen–Macaulay modules [4, 5]. Less is known if we consider non-commutative Cohen–Macaulay algebras. In [6] a criterion was given for a *primary* 1-dimensional Cohen–Macaulay algebra to be Cohen–Macaulay finite. In [1] (see also [3]) a criterion of Cohen–Macaulay finiteness is given for *normal* 2-dimensional Cohen–Macaulay algebras (maximal orders). As far as we know, there are no examples of 2-dimensional Cohen–Macaulay algebras which are not Cohen–Macaulay finite but are Cohen–Macaulay tame.

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In this paper we use the approach of Kahn [10] to study Cohen–Macaulay modules over normal non-commutative surface singularities. Just as in [10], we establish (in Section 2) a one-to-one correspondence between such modules and vector bundles over some, in general non-commutative, projective curves (Theorem 2.13). In Sections 3 and 4 we apply this result to a special case, which we call "*good elliptic*." It is analogous to the minimally elliptic case in [10], though seems somewhat too restrictive. Unfortunately, we could not find more general conditions which ensure such analogy. As an application, we present two examples of Cohen–Macaulay tame non-commutative surface singularities (Examples 4.1 and 4.2). We hope that this approach shall be useful in more general situations too.

1. Preliminaries

We fix an algebraically closed field $\mathbf{k}$, say *algebra* instead of $\mathbf{k}$-algebra, *scheme* instead of $\mathbf{k}$-scheme and write Hom and $\otimes$ instead of $\mathrm{Hom}_{\mathbf{k}}$ and $\otimes_{\mathbf{k}}$. We call a scheme X a *variety* if $\mathbf{k}(x) = \mathbf{k}$ for every closed point $x \in X$.

Definition 1.1. A *non-commutative scheme* is a pair (X, A), where X is a scheme and A is a sheaf of $\mathcal{O}_X$-algebras coherent as a sheaf of $\mathcal{O}_X$-modules. If X is a variety, (X, A) is called a *non-commutative variety.* We say that (X, A) is *affine, projective, excellent*, etc. if so is X.

A *morphism* of non-commutative schemes $(X, A) \to (Y, B)$ is their morphism as ringed spaces, i.e. a pair $(\varphi, \varphi^\sharp)$, where $\varphi : X \to Y$ is a morphism of schemes and $\varphi^\sharp : \varphi^{-1}A \to B$ is a morphism of sheaves of algebras. A morphism $(\varphi, \varphi^\sharp)$ is said to be *finite, projective* or *proper* if so is φ. We often omit $\varphi^\sharp$ and write $\varphi : (X, A) \to (Y, B)$.

For a non-commutative scheme (X, A) we denote by $\mathrm{Coh}\, A$ ($\mathrm{Qcoh}\, A$) the category of coherent (quasi-coherent) sheaves of A-modules. Every morphism $\varphi : (X, A) \to (Y, B)$ induces functors of direct image $\varphi_* : \mathrm{Qcoh}\, A \to \mathrm{Qcoh}\, B$ and inverse image $\varphi^* : \mathrm{Qcoh}\, B \to \mathrm{Qcoh}\, A$, where $\varphi^*\mathcal{F} = A \otimes_{\varphi^{-1}B} \varphi^{-1}\mathcal{F}$. Note that this inverse image does not coincide with the inverse image of sheaves of $\mathcal{O}_X$-modules. The latter (when used) will be denoted by φ^*_X. Note also that φ^* maps coherent sheaves to coherent. The pair (φ^*, φ_*) is a pair of adjoint functors, i.e. there is a functorial isomorphism $\mathrm{Hom}_A(\varphi^*\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_B(\mathcal{F}, \varphi_*\mathcal{G})$ for any sheaf of B-modules $\mathcal{F}$ and any sheaf of A-modules $\mathcal{G}$.

We call a coherent sheaf of A-modules $\mathcal{F}$ a *vector bundle* if it is locally projective, i.e. $\mathcal{F}_p$ is a projective A_p-module for every point $p \in X$. We denote by $\mathrm{VB}(A)$ the full subcategory of $\mathrm{Coh}\, A$ consisting of vector bundles.

A non-commutative scheme (X, A) is said to be *regular* if $\mathrm{gl.dim}\, A_p = \dim_p X$ for every point $p \in X$ (it is enough to check this property at the closed points).

We say that (X, A) is *reduced* if X is reduced and neither stalk A_p contains nilpotent ideals. Then, if $\mathcal{K} = \mathcal{K}_X$ is the sheaf of rational functions on X, $\mathcal{K}(A) = A \otimes_{\mathcal{O}_X} \mathcal{K}$ is a locally constant sheaf of semisimple $\mathcal{K}$-algebras. We

call it the *sheaf of rational functions* on (X, A). In this case each stalk A_p is an *order* in the algebra $\mathcal{K}(A)_p$, i.e. an $\mathcal{O}_{X,p}$-algebra finitely generated as $\mathcal{O}_{X,p}$-module and such that $\mathcal{K}_p A_p = \mathcal{K}(A)_p$. We say that (X, A) is *normal* if A_p is a maximal order in $\mathcal{K}(A)_p$ for each p. Note that a regular scheme is always reduced, but not necessarily normal.

A morphism $(\varphi, \varphi^\sharp) : (X, A) \to (Y, B)$ of reduced non-commutative schemes is said to be *birational* if $\varphi : X \to Y$ is birational and the induced map $\mathcal{K}(B) \to \mathcal{K}(A)$ is an isomorphism.

A *resolution* of a non-commutative scheme (X, A) is a proper birational morphism $(\pi, \pi^\sharp) : (\tilde{X}, \tilde{A}) \to (X, A)$, where $(\tilde{X}, \tilde{A})$ is regular and normal.

Remark 1.2. Let (X, A) be a non-commutative scheme and $C = \operatorname{cen}(A)$ be the center of A. (It means that $C_p = \operatorname{cen}(A_p)$ for every point $p \in X$.) Let also $X' = \operatorname{Spec} C$. The natural morphism $\varphi : X' \to X$ is finite and $A' = \varphi^{-1}A$ is a sheaf of $\mathcal{O}_{X'}$-modules, so we obtain a morphism $(\varphi, \varphi^\sharp) : (X', A') \to (X, A)$, where $\varphi^\sharp$ is identity. Moreover, the induced functors φ^* and φ_* define an equivalence of $\operatorname{Qcoh} A$ and $\operatorname{Qcoh} A'$. So, while we are interesting in study of sheaves, we can always suppose that A is a sheaf of *central* $\mathcal{O}_X$*-algebras*. Note that if (X, A) is normal and A is central, then X is also normal.

Given a non-commutative scheme (X, A) and a morphism of schemes $\varphi : Y \to X$, we can consider the non-commutative scheme $(Y, \varphi_Y^* A)$ and uniquely extend φ to the morphism $(Y, \varphi_Y^* A) \to (X, A)$ which we also denote by φ. Especially, if φ is a blow-up of a subscheme of X, we call the morphism $(Y, \varphi_Y^* A) \to (X, A)$ the blow-up of (X, A).

Definition 1.3. A reduced excellent non-commutative variety (X, A) is called a *non-commutative surface* if X is a surface, i.e. $\dim X = 2$. If $X = \operatorname{Spec} R$, where R is a local complete noetherian algebra with the residue field $\mathbf{k}$ (then it is automatically excellent), we say that (X, A) is a *germ of non-commutative surface singularity* or, for short, a *non-commutative surface singularity.* In what follows, we identify a non-commutative surface singularity (X, A) with the R-algebra $\Gamma(X, A)$ and the sheaves from $\operatorname{Qcoh} A$ with modules over this algebra (finitely generated for the sheaves from $\operatorname{Coh} A$).

If (X, A) is a non-commutative surface, there always is a normal non-commutative surface (X', A') and a finite birational morphism $\nu : (X', A') \to (X, A)$. We call (X', A'), as well as the morphism ν, a *normalization* of (X, A). Note that, unlike the commutative case, such normalization is usually not unique.

Let (X, A) be a connected central non-commutative surface such that X is normal, $C \subset X$ be an irreducible curve with the general point g, $\mathcal{K}_C(A) = A_g / \operatorname{rad} A_g$ and $\mathbf{k}_A(C) = \operatorname{cen} \mathcal{K}_C(A)$. A is normal if and only if it is Cohen–Macaulay (or, the same, reflexive) as a sheaf of $\mathcal{O}_X$-modules, $\mathcal{K}_C(A)$ is a simple algebra and $\operatorname{rad} A_g$ is a principal left (or right) A_g-ideal for every

curve C [12]. $\mathbf{k}_A(C)$ is a finite extension of the field of rational functions $\mathbf{k}(C) = \mathcal{O}_{X,g}/\operatorname{rad}\mathcal{O}_{X,g}$ on the curve C. The integer $e_C(A) = \dim_{\mathbf{k}(C)} \mathbf{k}_A(C)$ is called the *ramification index* of A on C, and A is said to be *ramified on* C if $e_C(A) > 1$. If p is a regular closed point of C, we denote by $e_{C,p}(A)$ the ramification index of the extension $\mathbf{k}_A(C)$ over $\mathbf{k}(C)$ with respect to the discrete valuation defined by the point p. For instance, if C is smooth, $e_{C,p}(A)$ is defined for all closed points $p \in C$. We denote by $D(A)$ the *ramification divisor* $D = D(A)$ which is the union of all curves $C \subset X$ such that A is ramified on C. Note that if $p \in X \setminus D(A)$, then A_p is an Azumaya algebra over $\mathcal{O}_{X,p}$.

Suppose that (X, A) is a normal non-commutative surface and A is central. Then X is Cohen–Macaulay and A is maximal Cohen–Macaulay as a sheaf of $\mathcal{O}_X$-modules. We denote by $\mathrm{CM}(A)$ the category of sheaves of maximal Cohen–Macaulay A-modules, i.e. the full subcategory of $\operatorname{Coh} A$ consisting of sheaves $\mathcal{F}$ which are maximal Cohen–Macaulay considered as sheaves of $\mathcal{O}_X$-modules. We often omit the attribute "maximal" and just say shortly "Cohen–Macaulay module." Obviously, $\mathrm{VB}(A) \subseteq \mathrm{CM}(A)$ and these categories coincide if and only if A is regular. For a sheaf $\mathcal{F} \in \operatorname{Coh} A$ we denote by $\mathcal{F}^\vee$ the sheaf $\mathcal{H}om_A(\mathcal{F}, A)$. It always belongs to $\mathrm{CM}(A)$. We also set $\mathcal{F}^\dagger = \mathcal{F}^{\vee\vee}$. There is a morphism of functors $\mathrm{Id} \to {}^\dagger$, which is isomorphism when restricted onto $\mathrm{CM}(A)$. If $\varphi : (X, A) \to (Y, B)$ is a morphism of central normal non-commutative surfaces, we set $\varphi^\dagger \mathcal{F} = (\varphi^* \mathcal{F})^\dagger$.

It is known that every non-commutative surface has a regular resolution. More precisely, we can use the following procedure of Chan–Ingalls [3].[1] The non-commutative surface (X, A) is said to be *terminal* [3, Definition 2.5] if the following conditions hold:

(1) X is smooth.
(2) All irreducible components of $D = D(A)$ are smooth.
(3) D only has normal crossings (i.e. nodes as singular points).
(4) At a node $p \in D$, for one component C_1 of D containing this point, the field $\mathbf{k}_A(C_1)$ is totally ramified over $\mathbf{k}(C_1)$ of degree $e = e_{C_1}(A) = e_{C_1,p}(A)$, and for the other component C_2 also $e_{C_2,p}(A) = e$.

It is shown in [3] that every terminal non-commutative surface is regular and every non-commutative surface (X, A) has a terminal resolution $\pi : (\tilde{X}, \tilde{A}) \to (X, A)$. Moreover, such resolution can be obtained by a sequence of morphisms π_i, where each π_i is either a blow-up of a closed point or a normalization. Then π is a projective morphism. If (X, A) is a normal non-commutative surface singularity, $\check{X} = X \setminus \{o\}$, where o is the unique closed point of X, the restriction of π onto $\pi^{-1}(\check{X})$ is an isomorphism and we always identify $\pi^{-1}(\check{X})$ with $\check{X}$. The subscheme $E = \pi^{-1}(o)_{\mathrm{red}}$ is a connected (though maybe reducible) projective curve called the *exceptional curve* of the resolution π.

[1] Note that the term "normal" is used in [3] in more wide sense, but we only need it for our notion of normality.

Recall also that, for a normal non-commutative surface singularity A, the category $\mathrm{CM}(A)$, as well as the ramification data of A, only depends on the algebra $\mathcal{K}(A)$ [1, (1.6)]. If A is central and *connected*, i.e. indecomposable as a ring, $\mathcal{K}(A)$ is a central simple algebra over the field $\mathcal{K}$, so the category $\mathrm{CM}(A)$ is defined by the class of $\mathcal{K}(A)$ in the Brauer group $\mathrm{Br}(\mathcal{K})$, and this class is completely characterized by its ramification data.

We also use the notion of *non-commutative formal scheme*, which is a pair $(\mathfrak{X}, \mathfrak{A})$, where $\mathfrak{X}$ is a "usual" (commutative) formal scheme and $\mathfrak{A}$ is a sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras coherent as a sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules. If (X, A) is non-commutative scheme and $Y \subset X$ is a closed subscheme, the *completion* $(\hat{X}, \hat{A})$ of (X, A) along Y is well-defined and general properties of complete schemes and their completions, as in [7, 9], hold in non-commutative case too.

2. Kahn's reduction

From now on we consider a normal non-commutative surface singularity (X, A) and suppose A central. We fix a resolution $\pi : (\tilde{X}, \tilde{A}) \to (X, A)$, where $\tilde{A}$ is also supposed central. Then $\mathrm{CM}(\tilde{A}) = \mathrm{VB}(\tilde{A})$ and we consider $\pi^\dagger$ as a functor $\mathrm{CM}(A) \to \mathrm{VB}(\tilde{A})$. A vector bundle $\mathcal{F}$ is said to be *full* if it is isomorphic to $\pi^\dagger M$ for some (maximal) Cohen–Macaulay A-module M. We denote by $\mathrm{VB}^f(\tilde{A})$ the full subcategory of $\mathrm{VB}(\tilde{A})$ consisting of full vector bundles. We also set $\omega_{\tilde{A}} = \mathcal{H}om_{\tilde{X}}(\tilde{A}, \omega_{\tilde{X}})$, where $\omega_{\tilde{X}}$ is a canonical sheaf over $\tilde{X}$, and call $\omega_{\tilde{A}}$ the *canonical sheaf* of $\tilde{A}$. It is locally free, i.e. belongs to $\mathrm{VB}(\tilde{A})$.

Given a coherent sheaf $\mathcal{F} \in \mathrm{Coh}\,\tilde{A}$, we denote by $\mathrm{ev}_{\mathcal{F}}$ the natural map $\Gamma(\tilde{X}, \mathcal{F}) \otimes \tilde{A} \to \mathcal{F}$.[2] We say that $\mathcal{F}$ is *globally generated* if $\mathrm{Im}\,\mathrm{ev}_{\mathcal{F}} = \mathcal{F}$ and *generically globally generated* if $\mathrm{supp}(\mathcal{F}/\mathrm{Im}\,\mathrm{ev}_{\mathcal{F}})$ is discrete, i.e. consists of finitely many closed points.

Theorem 2.1 (Cf. [10, Proposition 1.2]). (1) *The functor $\pi^\dagger$ establishes an equivalence between the categories $\mathrm{CM}(A)$ and $\mathrm{VB}^f(\tilde{A})$, its quasi-inverse being the functor π_*.*

(2) *A vector bundle $\mathcal{F} \in \mathrm{VB}(\tilde{A})$ is full if and only if the following conditions hold:*
   (a) *$\mathcal{F}$ is generically globally generated.*
   (b) *The restriction map $\Gamma(\tilde{X}, \mathcal{F}) \to \Gamma(\breve{X}, \mathcal{F})$ is surjective,*
   or equivalently, using local cohomologies,
   (b′) *The map $\alpha_{\mathcal{F}} : \mathrm{H}^1_E(\tilde{X}, \mathcal{F}) \to \mathrm{H}^1(\tilde{X}, \mathcal{F})$ is injective.*
   Under these conditions $\mathcal{F} \simeq \pi^\dagger \pi_* \mathcal{F}$.

Proof. Note that there is an exact sequence

$$0 \to \mathrm{tors}(\pi^* M) \to \pi^* M \xrightarrow{\gamma_M} \pi^\dagger M \to \overline{M} \to 0,$$

where $\mathrm{tors}(\mathcal{M})$ denotes the periodic part of $\mathcal{M}$ and the support of $\overline{M}$ consists of finitely many closed points. Since $\pi^* M$ is always globally generated, so

[2] Recall that $\Gamma(\tilde{X}, \mathcal{F}) \simeq \mathrm{Hom}_{\tilde{A}}(\tilde{A}, \mathcal{F})$.

is also $\operatorname{Im}\gamma_M$. Therefore, $\pi^\dagger M$ is generically globally generated. If M is Cohen–Macaulay, the restriction map $\Gamma(X,M)\to\Gamma(\check{X},M)=\Gamma(\check{X},\pi^*M)$ is an isomorphism. Since M naturally embeds into $\Gamma(\tilde{X},\pi^*M)$ and hence into $\Gamma(\tilde{X},\pi^\dagger M)$, the restriction $\Gamma(\tilde{X},\pi^\dagger M)\to\Gamma(\check{X},\pi^\dagger M)$ is surjective.

Suppose now that the conditions (a) and (b) hold. Set $M=\pi_*\mathcal{F}$. Since π is projective, $M\in\operatorname{Coh}A$. The condition (b) implies that $M\in\mathrm{CM}(A)$. Note that $\Gamma(X,M)=\Gamma(\tilde{X},\mathcal{F})$, so the image of the natural map $\pi^*M\to\mathcal{F}$ coincides with $\operatorname{Im}\gamma_M$. As $\mathcal{F}$ is generically globally generated, it implies that the natural map $\pi^\dagger M\to\mathcal{F}$ is an isomorphism. It proves (2).

Obviously, the functors $\pi^\dagger : CM(A)\to\mathrm{VB}^f(\tilde{A})$ and $\pi_* : \mathrm{VB}^f(\tilde{A})\to\mathrm{CM}(A)$ are adjoint. Moreover, if $M=\pi_*\mathcal{F}$, where $\mathcal{F}\in\mathrm{VB}^f(\tilde{A})$, there are functorial isomorphisms

$$\begin{aligned}\operatorname{Hom}_A(M,M)&\simeq\operatorname{Hom}_{\tilde{A}}(\pi^*\pi_*\mathcal{F},\mathcal{F})\simeq\\&\simeq\operatorname{Hom}_{\tilde{A}}(\pi^\dagger\pi_*\mathcal{F},\mathcal{F})\simeq\operatorname{Hom}_{\tilde{A}}(\mathcal{F},\mathcal{F}).\end{aligned}$$

It proves (1). □

Remark 2.2. A full vector bundle over $\tilde{A}$ need not be generically globally generated as a sheaf of $\mathcal{O}_{\tilde{X}}$-modules. Moreover, examples below show that even the sheaf $\tilde{A}=\pi^*A=\pi^\dagger A$ need not be generically globally generated as a sheaf of $\mathcal{O}_{\tilde{X}}$-modules.

Definition 2.3. From now on we consider a sheaf of ideals $\mathcal{I}$ in $\tilde{A}$ such that $\operatorname{supp}(\tilde{A}/\mathcal{I})\subseteq E$, $\Lambda=\tilde{A}/\mathcal{I}$ and $Z=\operatorname{Spec}(\operatorname{cen}\Lambda)$. Then (Z,Λ) is a projective non-commutative curve, i.e. a projective non-commutative variety of dimension 1 (maybe non-reduced). We set $\omega_Z=\mathcal{E}xt^1_{\tilde{X}}(\mathcal{O}_Z,\omega_{\tilde{X}})$ and

$$\omega_\Lambda=\mathcal{E}xt^1_{\tilde{A}}(\Lambda,\omega_{\tilde{A}})\simeq\mathcal{E}xt^1_{\tilde{X}}(\Lambda,\omega_{\tilde{X}})\simeq\mathcal{H}om_Z(\Lambda,\omega_Z).$$

The sheaves ω_Z and ω_Λ, respectively, are canonical sheaves for Z and Λ. It means that there are Serre dualities

$$\begin{aligned}\operatorname{Ext}^i_Z(\mathcal{F},\omega_Z)&\simeq\mathrm{DH}^{1-i}(E,\mathcal{F})\quad\text{for any }\mathcal{F}\in\operatorname{Coh}Z,\\\operatorname{Ext}^i_\Lambda(\mathcal{F},\omega_\Lambda)&\simeq\mathrm{DH}^{1-i}(E,\mathcal{F})\quad\text{for any }\mathcal{F}\in\operatorname{Coh}\Lambda,\end{aligned}$$

where $\mathrm{D}V$ denotes the vector space dual to V.

Definition 2.4. We say that an ideal I of a ring R is *bi-principal* if $I=aR=Ra$ for a non-zero-divisor $a\in R$. A sheaf of ideals $\mathcal{I}\subset\tilde{A}$ is said to be *locally bi-principal* if every point $x\in X$ has a neighbourhood U such that the ideal $\Gamma(U,\mathcal{I})$ is bi-principal in $\Gamma(U,\tilde{A})$.

Lemma 2.5. *If the sheaf of ideals $\mathcal{I}$ is locally bi-principal, then*

$$\omega_\Lambda\simeq\mathcal{H}om_{\tilde{A}}(\mathcal{I},\omega_{\tilde{A}})\otimes_{\tilde{A}}\Lambda.$$

Proof. Let $\mathcal{I}'=\mathcal{H}om_{\tilde{A}}(\mathcal{I},\omega_{\tilde{A}})$. Consider the locally free resolution $0\to\mathcal{I}\xrightarrow{\tau}\tilde{A}\to\Lambda\to0$ of Λ. Since $\omega_{\tilde{A}}$ is locally free over $\tilde{A}$, it gives an exact sequence

$$0\to\omega_{\tilde{A}}\xrightarrow{\tau^*}\mathcal{I}'\to\mathcal{E}xt^1_{\tilde{A}}(\Lambda,\omega_{\tilde{A}})\to0.$$

On the other hand, tensoring the same resolution with $\mathcal{I}'$ gives an exact sequence

$$0 \to \mathcal{I}' \otimes_{\tilde{A}} \mathcal{I} \xrightarrow{1\otimes\tau} \mathcal{I}' \to \mathcal{I}' \otimes_{\tilde{A}} \Lambda \to 0.$$

Since $\mathcal{I}$ is locally bi-principal, the natural map $\mathcal{I}' \otimes_{\tilde{A}} \mathcal{I} \to \omega_{\tilde{A}}$ is an isomorphism, and, if we identify $\mathcal{I}' \otimes_{\tilde{A}} \mathcal{I}$ with $\omega_{\tilde{A}}$, $1\otimes\tau$ identifies with τ^*. It implies the claim of the Lemma. □

Definition 2.6. Let $\mathcal{I} \subset \tilde{A}$ be a bi-principal sheaf of ideals such that $\operatorname{supp}(\tilde{A}/\mathcal{I}) = E$, $\Lambda = \tilde{A}/\mathcal{I}$ and $I = \mathcal{I}/\mathcal{I}^2$. (Note that $I \in \mathrm{VB}(\Lambda)$.) $\mathcal{I}$ is said to be a *weak reduction cycle* if

(1) I is generically globally generated as a sheaf of Λ-modules.
(2) $\mathrm{H}^1(E, I) = 0$.

If, moreover,

(3) $\omega_\Lambda^\vee = \mathcal{H}om_\Lambda(\omega_\Lambda, \Lambda)$ is generically globally generated over Λ,

$\mathcal{I}$ is called a *reduction cycle*.

For a weak reduction cycle $\mathcal{I}$ we define the *Kahn's reduction functor* $R_{\mathcal{I}} : \mathrm{CM}(A) \to \mathrm{VB}(\Lambda)$ as

$$R_{\mathcal{I}}(M) = \Lambda \otimes_{\tilde{A}} \pi^\dagger M.$$

We fix a weak reduction cycle $\mathcal{I}$ and keep the notation of the preceding Definition. We also set $\Lambda_n = \tilde{A}/\mathcal{I}^n$, $I_n = \mathcal{I}^n/\mathcal{I}^{n+1}$, $\mathcal{I}^{-n} = (\mathcal{I}^n)^\vee$ and $I_{-n} = \mathcal{I}^{-n}/\mathcal{I}^{1-n}$. In particular, $\Lambda_1 = \Lambda$ and $I_1 = I$. One easily sees that $I_n \simeq I \otimes_\Lambda I \otimes_\Lambda \dots \otimes_\Lambda I$ (n times) and $I_{-n} \simeq I_n^\vee = \mathcal{H}om_\Lambda(I_n, \Lambda)$.

Proposition 2.7. *If a coherent sheaf F of Λ-modules is generically globally generated, then $\mathrm{H}^1(E, I \otimes_\Lambda F) = 0$. In particular, $\mathrm{H}^1(E, I_n) = 0$*

Proof. Let $H = \Gamma(E, F)$. Consider the exact sequence

$$0 \to N \to H \otimes \Lambda \to F \to T \to 0,$$

where $N = \ker \mathrm{ev}_F$ and $\operatorname{supp} T$ is 0-dimensional. It gives the exact sequence

$$0 \to I \otimes_\Lambda N \to H \otimes I \to I \otimes_\Lambda F \to I \otimes_\Lambda T \to 0.$$

Since $\mathrm{H}^1(E, H \otimes I) = \mathrm{H}^1(E, I \otimes_\Lambda T) = 0$, we get that $\mathrm{H}^1(E, I \otimes_\Lambda F) = 0$. □

For any vector bundle $\mathcal{F}$ over $\tilde{A}$ set $F = \Lambda \otimes_{\tilde{A}} \mathcal{F}$ and $F_n = \Lambda_n \otimes_{\tilde{A}} \mathcal{F}$. There are exact sequences

$$\begin{gathered} 0 \to I_n \to \Lambda_{n+1} \to \Lambda_n \to 0, \\ 0 \to I_n \otimes_\Lambda F \to F_{n+1} \to F_n \to 0. \end{gathered} \tag{2.1}$$

For $n = 1$, tensoring the second one with $I^\vee = \mathcal{H}om_\Lambda(I, \Lambda)$, we get

$$0 \to F \to I^\vee \otimes_{\Lambda_2} F_2 \to I^\vee \otimes_\Lambda F \to 0.$$

Proposition 2.8. *Let $\mathcal{I}$ be a weak reduction cycle and $\mathcal{F}$ be a vector bundle over $\tilde{A}$ such that F is generically globally generated over Λ. Then $\mathcal{F}$ is also generically globally generated and $\mathrm{H}^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$.*

Note that if $\mathcal{F}$ is generically globally generated and $\mathrm{H}^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$, then F is also generically globally generated, since the map $\mathrm{H}^0(\tilde{X}, \mathcal{F}) \to \mathrm{H}^0(\tilde{X}, F)$ is surjective.

Proof. We first prove the second claim. Recall that, by the Theorem on Formal Functions [7, Theorem III.11.1],

$$\mathrm{H}^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) \simeq \varprojlim_n \mathrm{H}^1(E, \mathcal{I}/\mathcal{I}^n \otimes_{\tilde{A}} \mathcal{F}).$$

(We need not use completion, since $\mathrm{H}^1(\tilde{X}, \mathcal{M})$ is finite dimensional for every $\mathcal{M} \in \operatorname{Coh} \tilde{X}$.) Since $\mathcal{I}/\mathcal{I}^n$ is filtered by I_m $(1 \le m < n)$, we have to show that $\mathrm{H}^1(E, I_m \otimes_{\tilde{A}} \mathcal{F}) = \mathrm{H}^1(E, I_m \otimes_{\Lambda} F) = 0$ for all m. It follows from Proposition 2.7.

Note that $\Gamma(\tilde{X}, \mathcal{F}) = \Gamma(X, \pi_*\mathcal{F})$ and $\pi_*\mathcal{F}$ is globally generated, since X is affine. Moreover, the sheaves $\mathcal{F}$ and $\pi_*\mathcal{F}$ coincide on $\check{X}$. Hence $\Gamma(\tilde{X}, \mathcal{F})$ generate $\mathcal{F}_p$ for all $p \in \check{X}$. Therefore, we only have to prove that they generate $\mathcal{F}_p$ for almost all points $p \in E$. Since $\operatorname{supp} \Lambda = E$, it is enough to show that the global sections of $\mathcal{F}$ generate F_p for almost all $p \in E$. From the exact sequence $0 \to \mathcal{I} \otimes_{\tilde{A}} \mathcal{F} \to \mathcal{F} \to F \to 0$ and the equality $\mathrm{H}^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$ we see that the restriction $\Gamma(\tilde{X}, \mathcal{F}) \to \Gamma(E, F)$ is surjective. Since F is generically globally generated, so is also $\mathcal{F}$. $\square$

Corollary 2.9. *A locally bi-principal sheaf of ideals $\mathcal{I} \subset \tilde{A}$ is a weak reduction cycle if and only if*

(1) *$\mathcal{I}$ is generically globally generated.*
(2) $\mathrm{H}^1(\tilde{X}, \mathcal{I}) = 0$.

It is a reduction cycle if and only if, moreover, $\omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I}$ is generically globally generated.

Proof. If $\mathcal{I}$ is a weak reduction cycle, (1) and (2) follows from Proposition 2.8. Conversely, suppose that (1) and (2) hold. Since $\mathrm{H}^2(\tilde{X}, \_) = 0$, then $\mathrm{H}^1(E, I) = 0$. Moreover, just as in Proposition 2.7, $\mathrm{H}^1(\tilde{X}, \mathcal{I} \otimes_{\tilde{A}} \mathcal{F}) = 0$ for any generically globally generated $\mathcal{F}$. In particular, $\mathrm{H}^1(\tilde{X}, \mathcal{I}^2) = 0$. Hence the map $\Gamma(\tilde{X}, \mathcal{I}) \to \Gamma(E, I)$ is surjective, so I is generically globally generated.

Now let $\mathcal{I}$ be a weak reduction cycle. Note that, by Lemma 2.5,

$$\begin{aligned}\omega_{\Lambda}^{\vee} &= \mathcal{H}om_{\Lambda}(\mathcal{I}^{\vee} \otimes_{\tilde{A}} \omega_{\tilde{A}} \otimes_{\tilde{A}} \Lambda, \Lambda) \simeq \\ &\simeq \mathcal{H}om_{\tilde{A}}(\mathcal{I}^{\vee} \otimes_{\tilde{A}} \omega_{\tilde{A}}, \Lambda) \simeq \mathcal{H}om_{\tilde{A}}(\omega_{\tilde{A}}, \mathcal{I} \otimes_{\tilde{A}} \Lambda) \simeq \\ &\simeq \omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I} \otimes_{\tilde{A}} \Lambda \simeq \omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I}/\omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I}^2.\end{aligned}$$

Hence, by Proposition 2.8, $\omega_{\Lambda}^{\vee}$ is generically globally generated if and only if so is $\omega_{\tilde{A}}^{\vee} \otimes_{\tilde{A}} \mathcal{I}$. $\square$

Proposition 2.10. *A reduction cycle always exists.*

Proof. Since the intersection form is negative definite on the group of divisors on $\tilde{X}$ with support E [11], there is a divisor D with support E such that $\mathcal{O}_{\tilde{X}}(-D)$ is ample. Therefore, for some $n > 0$, $\mathcal{I} = \tilde{A}(-nD)$ as well as $\omega_{\tilde{A}}^{\vee}(-nD)$ are generically globally generatedand, moreover, $\mathrm{H}^1(\tilde{X}, \mathcal{I}) = 0$. Obviously, $\mathcal{I}$ is bi-principal, so it is a reduction cycle. $\square$

Now we need the following modification of the Wahl's lemma [13, Lemma B.2].

Lemma 2.11. *If $\mathcal{F}$ is a vector bundle over $\tilde{A}$, then*

$$\mathrm{H}^1_E(\tilde{X}, \mathcal{F}) \simeq \varinjlim_n \mathrm{H}^0(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n)$$

Moreover, the natural homomorphisms

$$\mathrm{H}^0(E, \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n) \to \mathrm{H}^0(E, \mathcal{I}^{-n-1} \otimes_{\tilde{A}} F_{n+1})$$

are injective.

Proof. Note that $\mathrm{H}^1_E(\tilde{X}, \mathcal{F}) \simeq \varinjlim_n \mathrm{Ext}^1_{\tilde{A}}(\Lambda_n, \mathcal{F})$. Consider the spectral sequence $\mathrm{H}^p(\tilde{X}, \mathcal{E}xt^q_{\tilde{A}}(\Lambda_n, \mathcal{F})) \Rightarrow \mathrm{Ext}^{p+q}_{\tilde{A}}(\Lambda_n, \mathcal{F})$. Since $\mathcal{H}om_{\tilde{A}}(\Lambda_n, \mathcal{F}) = 0$, the exact sequence of the lowest terms gives an isomorphism $\mathrm{Ext}^1_{\tilde{A}}(\Lambda_n, \mathcal{F}) \simeq \mathrm{H}^0(E, \mathcal{E}xt^1_{\tilde{A}}(\Lambda_n, \mathcal{F}))$. Applying $\mathcal{H}om_{\tilde{A}}(\_, \mathcal{F})$ to the exact sequence $0 \to \mathcal{I}^n \to \tilde{A} \to \Lambda_n \to 0$, we get the exact sequence

$$0 \to \mathcal{F} = \tilde{A} \otimes_{\tilde{A}} \mathcal{F} \to \mathcal{H}om_{\tilde{A}}(\mathcal{I}^n, \mathcal{F}) \simeq \mathcal{I}^{-n} \otimes_{\tilde{A}} \mathcal{F} \to \mathcal{E}xt^1_{\tilde{A}}(\Lambda_n, \mathcal{F}) \to 0,$$

whence $\mathcal{E}xt^1_{\tilde{A}}(\Lambda_n, \mathcal{F}) \simeq (\mathcal{I}^{-n}/\tilde{A}) \otimes_{\tilde{A}} \mathcal{F}$. Moreover, since $\mathcal{I}^{-n}/\tilde{A} \subseteq \mathcal{I}^{-n-1}/\tilde{A}$ and $\mathcal{F}$ is locally projective, we get an embedding $(\mathcal{I}^{-n}/\tilde{A}) \otimes_{\tilde{A}} \mathcal{F} \hookrightarrow (\mathcal{I}^{-n-1}/\tilde{A}) \otimes_{\tilde{A}} \mathcal{F}$, hence an embedding of cohomologies. It remains to note that

$$(\mathcal{I}^{-n}/\tilde{A}) \otimes_{\tilde{A}} \mathcal{F} \simeq (\mathcal{I}^{-n}/\tilde{A}) \otimes_{\Lambda_n} F_n \simeq \mathcal{I}^{-n} \otimes_{\tilde{A}} F_n,$$

since $\mathcal{I}^n$ annihilates $\mathcal{I}^{-n}/\tilde{A}$. $\square$

Since $I \otimes_{\Lambda} F \simeq \mathcal{I} \otimes_{\tilde{A}} F$, there is an exact sequence

$$0 \to \mathcal{I} \otimes_{\tilde{A}} F \to F_2 \to F \to 0.$$

Multiplying it with $\mathcal{I}^{\vee}$, we get an exact sequence

$$0 \to F \to \mathcal{I}^{\vee} \otimes_{\tilde{A}} F_2 \to \mathcal{I}^{\vee} \otimes_{\tilde{A}} F \to 0, \tag{2.2}$$

which gives the coboundary map $\theta_F : \mathrm{H}^0(E, \mathcal{I}^{\vee} \otimes_{\tilde{A}} F) \to \mathrm{H}^1(E, F)$.

Proposition 2.12 (Cf. [10, Proposition 1.6])**.** *Let $\mathcal{I}$ be a weak reduction cycle. A vector bundle $\mathcal{F} \in \mathrm{VB}(\tilde{A})$ is full if and only if*

(1) *F is generically globally generated over Λ.*
(2) *The coboundary map θ_F is injective.*

Proof. Let $\mathcal{F}$ be generically globally generated. Since $\mathrm{H}^1(\tilde{X},\mathcal{I})=0$, also $\mathrm{H}^1(\tilde{X},\mathcal{I}\otimes_{\tilde{A}}\mathcal{F})=0$. Therefore, the map $\Gamma(\tilde{X},\mathcal{F})\to\Gamma(E,F)$ is surjective, so F is generically globally generated. Conversely, if F is generically globally generated, so is $\mathcal{F}$ by Proposition 2.8. Hence, this condition (1) is equivalent to the condition (1) of Proposition 2.1. So now we suppose that both $\mathcal{F}$ and F are generically globally generated.

Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^1_E(\tilde{X},\mathcal{F}) & \xrightarrow{\alpha_{\mathcal{F}}} & \mathrm{H}^1(\tilde{X},\mathcal{F})\\
{\scriptstyle i}\Big\uparrow & & \Big\downarrow{\scriptstyle p}\\
\mathrm{H}^0(E,\mathcal{I}^\vee\otimes_{\tilde{A}}F) & \xrightarrow{\theta_F} & \mathrm{H}^1(E,F)
\end{array}$$

Here i is an embedding from Lemma 2.11 and p is an isomorphism, since $\mathrm{H}^1(\tilde{X},\mathcal{I}\otimes_{\tilde{A}}\mathcal{F})=0$. If $\mathcal{F}$ is full, $\alpha_{\mathcal{F}}$ is injective, hence so is θ_F.

Conversely, suppose that θ_F is injective. We show that all embeddings

$$\mathrm{H}^0(E,\mathcal{I}^{-n}\otimes_{\tilde{A}}F_n)\to\mathrm{H}^0(E,\mathcal{I}^{-n-1}\otimes_{\tilde{A}}F_{n+1}) \tag{2.3}$$

from Lemma 2.11 are actually isomorphisms. It implies that $\alpha_{\mathcal{F}}$ is injective, so $\mathcal{F}$ is full.

The map (2.3) comes from the exact sequence

$$0\to\mathcal{I}^{-n}\otimes_{\tilde{A}}F_n\to\mathcal{I}^{-n-1}\otimes_{\tilde{A}}F_{n+1}\to\mathcal{I}^{-n-1}\otimes_{\tilde{A}}F\to 0 \tag{2.4}$$

obtained from the exact sequence

$$0\to\mathcal{I}\otimes_{\tilde{A}}F_n\to F_{n+1}\to F\to 0$$

by tensoring with $\mathcal{I}^{-n-1}$. So we have to show that the connecting homomorphism

$$\beta_n:\mathrm{H}^0(E,\mathcal{I}^{-n-1}\otimes_{\tilde{A}}F)\to\mathrm{H}^1(E,\mathcal{I}^{-n}\otimes_{\tilde{A}}F_n)$$

is injective. We actually prove that even the map

$$\beta_n':\mathrm{H}^0(E,\mathcal{I}^{-n-1}\otimes_{\tilde{A}}F)\to\mathrm{H}^1(E,\mathcal{I}^{-n}\otimes_{\tilde{A}}F),$$

which is the composition of β_n with the natural map $\mathrm{H}^1(E,\mathcal{I}^{-n-1}\otimes_{\tilde{A}}F_n)\to\mathrm{H}^1(E,\mathcal{I}^{-n}\otimes_{\tilde{A}}F)$ is injective.

Indeed, β_0 coincides with θ_F. Since all sheaves $\mathcal{I}^n$ are generically globally generated, there is a homomorphism $m\tilde{A}\to\mathcal{I}^n$ whose cokernel has a finite support. Taking duals, we get an embedding $\mathcal{I}^{-n}\hookrightarrow m\tilde{A}$. Tensoring this embedding with the exact sequence (2.4) for $n=0$ and taking cohomologies, we get a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^0(E,\mathcal{I}^{-n-1}\otimes_{\tilde{A}}F) & \xrightarrow{\beta_n'} & \mathrm{H}^1(E,\mathcal{I}^{-n}\otimes_{\tilde{A}}F)\\
\Big\downarrow & & \Big\downarrow\\
m\mathrm{H}^0(E,\mathcal{I}^{-1}\otimes_{\tilde{A}}F) & \longrightarrow & m\mathrm{H}^1(E,F)
\end{array}$$

where the second horizontal and the first vertical maps are injective. Therefore, β_n' is injective too, which accomplishes the proof. $\square$

We call a vector bundle $F \in \mathrm{VB}(\Lambda)$ *full* if $F \simeq \Lambda \otimes_{\tilde{A}} \mathcal{F}$, where $\mathcal{F}$ is a full vector bundle over $\tilde{A}$.

Theorem 2.13 (Cf. [10, Theorem 1.4])**.** *Let $\mathcal{I}$ be a weak reduction cycle. A vector bundle $F \in \mathrm{VB}(\Lambda)$ is full if and only if*

1. *F is generically globally generated.*
2. *There is a vector bundle $F_2 \in \mathrm{VB}(\Lambda_2)$ such that $\Lambda \otimes_{\tilde{A}} F_2 \simeq F$ and the connecting homomorphism $\theta_F : \mathrm{H}^0(E, \mathcal{I}^\vee \otimes_{\tilde{A}} F) \to \mathrm{H}^1(E, F)$ coming from the exact sequence* (2.2) *is injective.*

If, moreover, $\mathcal{I}$ is a reduction cycle, the full vector bundle $\mathcal{F} \in \mathrm{VB}(\tilde{A})$ such that $\Lambda \otimes_{\tilde{A}} \mathcal{F} \simeq F$ is unique up to isomorphism. Thus the reduction functor $R_{\mathcal{I}}$ induces a one-to-one correspondence between isomorphism classes of Cohen–Macaulay A-modules and isomorphism classes of full vector bundles over Λ.

Proof. Let $\mathcal{I}$ be a weak reduction cycle, F satisfies (1) and (2). If $U \subset E$ is an affine open subset, there is an exact sequence

$$0 \to I_n(U) \to \Lambda_{n+1}(U) \to \Lambda_n(U) \to 0,$$

where the ideal $I_n(U)$ is nilpotent (actually, $I_n(U)^2 = 0$). Therefore, given a projective $\Lambda_n(U)$-module P_n, there is a projective $\Lambda_{n+1}(U)$-module P_{n+1} such that $\Lambda_n(U) \otimes_{\Lambda_{n+1}(U)} P_{n+1} \simeq P_n$. Moreover, if P'_n is another projective $\Lambda_n(U)$-module, P'_{n+1} is a projective $\Lambda_{n+1}(U)$-module such that $\Lambda_n(U) \otimes_{\Lambda_{n+1}(U)} P'_{n+1} \simeq P'_n$ and $\varphi_n : P_n \to P'_n$ is a homomorphism, it can be lifted to a homomorphism $\varphi_{n+1} : P_{n+1} \to P'_{n+1}$, and if φ_n is an isomorphism, so is φ_{n+1} too.

Consider an affine open cover $E = U_1 \cup U_2$. Let $P_{2,i} = F_2(U_i)$. Iterating the above procedure, we get projective $\Lambda_n(U_i)$-modules $P_{n,i}$ such that

$$\Lambda_n(U_i) \otimes_{\Lambda_{n+1}(U_i)} P_{n+1,i} \simeq P_{n,i}$$

for all $n \geq 2$. If $U = U_1 \cap U_2$, there is an isomorphism $\varphi_2 : P_{2,1}(U) \xrightarrow{\sim} P_{2,2}(U)$. It can be lifted to $\varphi_n : P_{n,1}(U) \xrightarrow{\sim} P_{n,2}(U)$ so that the restriction of φ_{n+1} to $P_{n,1}$ coincides with φ_n. Hence there are vector bundles F_n over Λ_n such that $\Lambda_n \otimes_{\tilde{A}} F_{n+1} \simeq F_n$. Taking inverse image, we get a vector bundle $\hat{\mathcal{F}} = \varprojlim_n F_n$ over the formal non-commutative scheme $(\hat{X}, \hat{A})$ which is the completion of $(\tilde{X}, \tilde{A})$ along the subscheme E. As $\tilde{X}$ is projective, hence proper over X, $\hat{\mathcal{F}}$ uniquely arises as the completion of a vector bundle $\mathcal{F}$ over $\tilde{A}$ such that $\Lambda_n \otimes_{\tilde{A}} \mathcal{F} \simeq F_n$ for all n (see [9, Theorem 5.1.4]). If we choose F_2 so that the condition (2) holds, $\mathcal{F}$ is full by Proposition 2.12. Thus F is full as well.

Let now $\mathcal{I}$ be a reduction cycle, F be a full vector bundle over Λ and F_n be vector bundles over Λ_n such that $\Lambda_n \otimes_{\tilde{A}} F_{n+1} \simeq F_n$ for all n and F_2 satisfies the condition (2). As we have already mentioned, all choices of F_n are locally isomorphic. Therefore, if we fix one of them, their isomorphism classes are in one-to-one correspondence with the cohomology set $\mathrm{H}^1(E, \mathcal{A}ut\, F_n)$ [8].

From the exact sequence (2.1) we obtain an exact sequence of sheaves of groups

$$0 \to \mathcal{H} \to \mathcal{A}ut\, F_{n+1} \xrightarrow{\rho} \mathcal{A}ut\, F_n \to 0,$$

where $\mathcal{H} = \ker \rho \simeq \mathcal{H}om_{\Lambda_n}(F_n, I_n \otimes_\Lambda F) \simeq \mathcal{H}om_\Lambda(F, I_n \otimes_\Lambda F)$. It gives an exact sequence of cohomologies

$$\begin{aligned} 0 \to \mathrm{Hom}_\Lambda(F_n, I_n \otimes_\Lambda F) \to \mathrm{Aut}\, F_{n+1} \to \mathrm{Aut}\, F_n \xrightarrow{\delta} \\ \to \mathrm{Ext}^1_\Lambda(F, I_n \otimes_\Lambda F) \to \mathrm{H}^1(E, \mathcal{A}ut\, F_{n+1}) \to \mathrm{H}^1(E, \mathcal{A}ut\, F_n). \end{aligned}$$

The isomorphism classes of liftings F_{n+1} of a given F_n are in one-to-one correspondence with the orbits of the group $\mathrm{Aut}\, F_n$ naturally acting on $\mathrm{Ext}^1_\Lambda(F, I_n \otimes_\Lambda F)$. [8, Proposition 5.3.1].

We write automorphisms of F_n in the form $1 + \varphi$ for $\varphi \in \mathrm{End}\, F_n$. Then $\delta(1 + \varphi) = \delta_n(\varphi)$, where $\delta_n : \mathrm{Hom}_{\Lambda_n}(F_n, F_n) \to \mathrm{Ext}^1_\Lambda(F, I_n \otimes_\Lambda F)$ is the connecting homomorphism coming from the exact sequence (2.1). We restrict δ_n to $\mathrm{Hom}_\Lambda(F, I_{n-1} \otimes_\Lambda F)$ (see the same exact sequence, with n replaced by $n-1$). The resulting homomorphism $\delta'_n : \mathrm{Hom}_\Lambda(F, I_{n-1} \otimes_\Lambda F) \to \mathrm{Ext}^1_\Lambda(F, I_n \otimes_\Lambda F)$ coincides with the connecting homomorphism coming from the exact sequence (2.2) tensored with $\mathcal{I}^{n-1}$.

Claim 1. *δ'_n is surjective.*

Indeed, since F, I_{n-1} and $\omega^\vee_\Lambda$ are generically globally generated, so is their tensor product. Hence, there is a homomorphism $m\Lambda \to \omega^\vee_\Lambda \otimes_\Lambda I_{n-1} \otimes_\Lambda F$, thus also $m\omega_\Lambda \to I_{n-1} \otimes_\Lambda F$ whose cokernel has discrete support. Applying $\mathrm{Hom}_\Lambda(F, \_)$, we get a commutative diagram

$$\begin{tikzcd} m\,\mathrm{Hom}_\Lambda(F, \omega_\Lambda) \arrow[r] \arrow[d] & m\,\mathrm{Ext}^1_\Lambda(F, I \otimes_\Lambda \omega_\Lambda) \arrow[d, "\eta"'] \\ \mathrm{Hom}_\Lambda(F, I_{n-1} \otimes_\Lambda F) \arrow[r, "\delta'_n"] & \mathrm{Ext}^1_\Lambda(F, I_n \otimes_\Lambda F), \end{tikzcd}$$

where η is surjective. Note that the first horizontal map here is the m-fold Serre dual θ^*_F to the map

$$\theta_F : \mathrm{Hom}_\Lambda(I, F) \simeq \mathrm{H}^0(E, I^\vee \otimes_\Lambda F) \to \mathrm{Ext}^1_\Lambda(\Lambda, F) \simeq \mathrm{H}^1(E, F),$$

which is injective. Therefore, θ^*_F is surjective and so is also δ'_n.

If $n > 1$, every homomorphism $1 + \varphi$ with $\varphi \in \mathrm{Hom}_\Lambda(F, I_{n-1} \otimes_\Lambda F)$ is invertible. Hence δ is surjective and F_{n+1} is unique up to isomorphism. If $n = 1$, the set $\{\, \varphi \in \mathrm{Hom}_\Lambda(F, F) \mid 1 + \varphi \text{ is invertible} \,\}$ is open in $\mathrm{Aut}\, F$. Therefore, its image in $\mathrm{Ext}^1_\Lambda(F, I \otimes_\Lambda F)$ is an open orbit of $\mathrm{Aut}\, F$. If we choose another lifting F'_2 of F so that the condition (2) holds, it also gives an open orbit. Since there can be at most one open orbit, they coincide, hence $F'_2 \simeq F_2$. Now, if $\mathcal{F}$ and $\mathcal{F}'$ are two full vector bundles over $\tilde{A}$ such that $\Lambda \otimes_{\tilde{A}} \mathcal{F} \simeq \Lambda \otimes_{\tilde{A}} \mathcal{F}' \simeq F$, we can glue isomorphisms $\Lambda_n \otimes_{\tilde{A}} \mathcal{F} \xrightarrow{\sim} \Lambda_n \otimes_{\tilde{A}} \mathcal{F}'$ into an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}'$. □

Claim 1 also implies the following result.

Corollary 2.14 (Cf. [10, Corollary 1.10]). *If* $\mathcal{F} \in \mathrm{VB}^f(\tilde{A})$, *then* $\mathrm{Ext}^1_{\tilde{A}}(\mathcal{F},\mathcal{F}) \simeq \mathrm{Ext}^1_\Lambda(F,F)$.

We omit the proof since it just copies that from [10].

3. Good elliptic case

There is one special case when the conditions of Theorem 2.13 can be made much simpler. It is analogous to the case of *minimally elliptic* surface singularities considered in [10, Section 2]. We are not aware of the full generality when it can be done, so we only confine ourselves to a rather restricted situation. Thus the following definition shall be considered as very preliminary. It will be used in the examples studied in the next section.

Definition 3.1. Let $\pi : (\tilde{X},\tilde{A}) \to (X,A)$ be a resolution of a non-commutative surface singularity, $\mathcal{I}$ be a weak reduction cycle and $\Lambda = \tilde{A}/\mathcal{I}$. We say that the weak reduction cycle $\mathcal{I}$ is *good elliptic* if $\Lambda \simeq \mathcal{O}_Z$ where Z is a reduced curve of arithmetic genus 1 (hence $\omega_Z \simeq \mathcal{O}_Z$). Obviously, then $\mathcal{I}$ is a reduction cycle. If a non-commutative surface singularity (X,A) has a resolution $(\tilde{X},\tilde{A})$ such that there is a good elliptic reduction cycle $\mathcal{I} \subset \tilde{A}$, we say that (X,A) is a *good elliptic non-commutative surface singularity.*

Remark 3.2. One easily sees that being good elliptic is equivalent to fulfilments of the following conditions for some resolution:

(1) $\mathrm{h}^1(\tilde{X},\tilde{A}) = 1$.
(2) There is a weak reduction cycle $\mathcal{I}$ such that Λ is commutative and reduced.

Then $\mathcal{I}$ is also a reduction cycle.

For good elliptic non-commutative surface singularity we can state a complete analogue of [10, Theorem 2.1]. Moreover, the proof is just a copy of the Kahn's proof, so we omit it.

Theorem 3.3. *Suppose that* $\mathcal{I}$ *is a good elliptic reduction cycle for a resolution* $(\tilde{X},\tilde{A})$ *of a non-commutative surface singularity* (X,A), $\Lambda = \tilde{A}/\mathcal{I}$ *and* $I = \mathcal{I}/\mathcal{I}^2$. *A vector bundle* F *over* Λ *is full if and only if* $F \simeq G \oplus m\Lambda$, *where the following conditions hold:*

(1) G *is generically globally generated.*
(2) $\mathrm{H}^1(E,G) = 0$.
(3) $m \geq \mathrm{h}^0(E, I^\vee \otimes_\Lambda G)$.[3]

If these conditions hold and M *is the Cohen–Macaulay* A*-module such that* $F \simeq R_{\mathcal{I}}M$, *then* M *is indecomposable if and only if either* $m = \mathrm{h}^0(E, I^\vee \otimes_\Lambda G)$ *or* $F = \Lambda$ *(then* $M = A$*).*

Now, just as in [5] (and with the same proof), we obtain the following result.

[3] If we identify Λ with $\mathcal{O}_Z$, then $I^\vee \otimes_\Lambda G$ is identified with $G(Z)$.

Corollary 3.4. *Suppose that $\mathcal{I}$ is a good elliptic reduction cycle for a resolution $(\tilde{X},\tilde{A})$ of a non-commutative surface singularity (X,A) and $\Lambda=\tilde{A}/\mathcal{I}\simeq\mathcal{O}_Z$. The non-commutative surface singularity (X,A) is Cohen–Macaulay tame if and only if Z is either a smooth elliptic curve or a Kodaira cycle* (*a* cyclic configuration *in the sense of* [5])*. Otherwise it is Cohen–Macaulay wild.*

For the definitions of Cohen–Macaulay tame and wild singularities see [5, Section 4]. Though in this paper only the commutative case is considered, the definitions are completely the same in the non-commutative one.

4. Examples

In what follows we consider non-commutative surface singularities (X,A), where $X=\operatorname{Spec}R$ and $R=\mathbf{k}[[u,v]]$. We define A by generators and relations. The ramification divisor $D=D(A)$ is then given by one relation $F=0$ for some $F\in R$, so it is a *plane curve singularity.*

When blowing up the closed point o, we get the subset $\tilde{X}\subseteq\operatorname{Proj}R[\alpha,\beta]$ given by the equation $u\beta=v\alpha$. We cover it by the affine charts $U_1:\ \beta\neq0$ and $U_2:\ \alpha\neq0$, so their coordinate rings are, respectively, $R_1=R[\xi]/(u-\xi v)$ and $R_2=R[\eta]/(v-\eta u)$, where $\xi=\alpha/\beta$ and $\eta=\beta/\alpha$.

Example 4.1.

$$A=R\langle\, x,y\mid x^2=v,\ y^2=u(u^2+\lambda v^2),\ xy+yx=2\varepsilon uv\,\rangle,$$

where $\lambda\notin\{\,0,1\,\}$ and $\varepsilon^2=1+\lambda$. Then $F=uv(u-v)(u-\lambda v)$, so D is of type T_{44}. We set $z=xy$, so $\{\,1,x,y,z\,\}$ is an R-basis of A and $z^2=2\varepsilon uvz-uv(u^2+\lambda v^2)$. One can check that $\mathbf{k}_C(A)$ is a field, namely a quadratic extension of $\mathbf{k}(C)$, for every component of D. For instance, if this component is $u=v$, and g is its general point, then, modulo the ideal $(u-v)A_g$, $(z-\varepsilon uv)^2=0$, so $z-\varepsilon uv\in\operatorname{rad}A_g$. Moreover,

$$\begin{aligned}(z-\varepsilon uv)^2&=z^2-2\varepsilon uvz+(1+\lambda)u^2v^2=\\&=-uv(u^2+\lambda v^2)+(1+\lambda)u^2v^2=\\&=uv(u-v)(\lambda v-u).\end{aligned}$$

Since $uv(\lambda v-u)$ is invertible in A_g, $u-v\in(z-\varepsilon uv)A_g$. One easily sees that $(z-\varepsilon uv)A_g$ is a two-sided ideal and $A_g/(z-\varepsilon uv)A_g\simeq\mathbf{k}[[u]][x]/(x^2-u)$ is a field. (Note that in this factor $\varepsilon uvx=zx=vy$, so $y=\varepsilon ux$.) Therefore, (X,A) is normal and its ramification index equals 2 on every component of D.

After blowing up the closed point $o\in X$, we get

$$\pi^*A(U_1)\simeq R_1\langle\, x,y\mid x^2=v,\ y^2=\xi(\xi^2+\lambda)v^3,\ xy+yx=2\varepsilon\xi v^2\,\rangle$$

and $z^2=2\varepsilon\xi v^2z-\xi v^4(\xi^2+\lambda)$. So we can consider the R_1-subalgebra $A_1=\pi^*A(U_1)\langle\, z_1\,\rangle$ of $\mathcal{K}(A)$, where $z_1=v^{-2}z-\varepsilon\xi$. Note that $y_1=v^{-1}y=$

$xz_1 + \varepsilon\xi x \in A_1$.

$$\pi^*A(U_2) \simeq R_2\langle\, x, y \mid x^2 = \eta u,\ y^2 = u^3(1+\lambda\eta^2),\ xy + yx = 2\varepsilon\eta u^2 \,\rangle$$

and $z^2 = 2\varepsilon\eta u^2 z - \eta u^4(1+\eta^2\lambda)$. So we can consider the R_2-subalgebra $A_2 = \pi^*A(U_2)\langle\, y_2, z_2 \,\rangle$ of $\mathcal{K}(A)$, where $y_2 = u^{-1}y$, $z_2 = u^{-2}z - \varepsilon\eta$.

Since $y_2 = \eta y_1$ and $z_2 = \eta^2 z_1$, $A_1(U_1 \cap U_2) = A_2(U_1 \cap U_2)$, so we can consider the non-commutative surface $(\tilde{X}, \tilde{A})$, where $\tilde{A}(U_1) = A_1$, $\tilde{A}(U_2) = A_2$. One can check, just as above, that it is normal. Its ramification divisor $\tilde{D}$ is given on U_1 by the equation $\xi v(\xi - 1)(\xi - \lambda) = 0$ and on U_2 by $u\eta(1-\eta)(1-\lambda\eta) = 0$, so its components are projective lines and have normal crossings. Moreover, $e_C(A) = 2$ for every component C of $\tilde{D}$, and if $x \in C$ is a node of $\tilde{D}$, then $e_{C,x}(A) = 2$. Hence $(\tilde{X}, \tilde{A})$ is a terminal resolution of (X, A).

Consider the ideal $\mathcal{I} \subset \tilde{A}$ such that $\mathcal{I}(U_1) = (x)$ and $\mathcal{I}(U_2) = (x, y_2)$. Note that $\eta y_2 = xz_2 - \varepsilon\eta x$, $z_2 y_2 = (1+\lambda\eta^2)x - \varepsilon\eta y_2$ and $y_2^2 = (1+\lambda\eta^2)u$. Therefore, if $p \in U_2$ and $\eta(p) \neq 0$, then $y_2, u \in \tilde{A}_p x = x\tilde{A}_p$, while if $p \in U_2$ and $\eta(p) = 0$, then $x, u \in \tilde{A}_p y_2 = y_2\tilde{A}_p$. Hence $\mathcal{I}$ is bi-principal.

$$A_1/\mathcal{I}(U_1) \simeq \mathbf{k}[\xi, z_1]/(z_1^2 - \xi(\xi-1)(\lambda-\xi)),$$

and

$$A_2/\mathcal{I}(U_2) \simeq \mathbf{k}[\eta, z_2]/(z_2^2 - \eta(1-\eta)(\lambda\eta-1)),$$

hence $\Lambda = \tilde{A}/\mathcal{I} \simeq \mathcal{O}_Z$, where Z is an elliptic curve. Moreover, x is a global section of $\mathcal{I}$, hence of $I = \mathcal{I}/\mathcal{I}^2$, and it generates I_p for every point $p \in Z$ except the point ∞ on the chart U_2, where $\eta = 0$. So $\mathcal{I}$ is a good elliptic reduction cycle and $I \simeq \mathcal{O}_Z(\infty)$.

Now, by Theorem 3.3, Cohen–Macaulay modules over A can be obtained as follows. We identify Z with $\mathrm{Pic}^0(Z)$ taking ∞ as the zero point. Denote by $G(r,d;p)$ the indecomposable vector bundle over Z of rank r, degree d and the Chern class $p \in Z = \mathrm{Pic}^0(Z)$ (see [2]). It is generically globally generated if and only if either $d > 0$ or $d = 0$, $r = 1$ and $p = \infty$. In the latter case $G(1,0;\infty) \simeq \mathcal{O}_Z$. Then $I^\vee \otimes_\Lambda G(r,d;p) \simeq G(r, d-r; p)$. Moreover,

$$\mathrm{h}^0(Z, G(r,d;p)) = \begin{cases} 0 & \text{if } 1 \le d < 0 \text{ or } d = 0 \text{ and } p \neq \infty, \\ 1 & \text{if } d = 0 \text{ and } p = \infty, \\ d & \text{if } d > 0. \end{cases}$$

So if M is an indecomposable Cohen–Macaulay A-module and $M \not\simeq A$, then it is uniquely determined by its Kahn reduction $R_{\mathcal{I}}M$ which is one of the following vector bundles:

$G(r,d;p)$, where $d < r$ or $d = r$, $p \neq \infty$; then $\mathrm{rk}\, M = r$.
$G(r,r;\infty) \oplus \mathcal{O}_Z$, where $r > 1$; then $\mathrm{rk}\, M = r + 1$.
$G(r,d;p) \oplus (d-r)\mathcal{O}_Z$, where $d > r$; then $\mathrm{rk}\, M = d$.

In particular, A is *Cohen–Macaulay tame* in the sense of [5]. Namely, for a fixed rank r, Cohen–Macaulay A-modules of rank r, except one of them, form

$2(r-1)$ families parametrized by Z and one family parametrized by $Z\setminus\{\infty\}$, arising, respectively, from $G(d,r,p)$ $(1\le d<r)$, $G(r',r,p)$ $(1\le r'<r)$ and $G(r,r,p)$ $(p\ne\infty)$.

Example 4.2.

$$A=R\langle\, x,y \mid x^3=v,\ y^3=u(u-v),\ xy=\zeta yx\,\rangle,\ \text{where } \zeta^3=1,\ \zeta\ne 1.$$

Then $F=uv(u-v)$ (the singularity of type D_4). Just as above, one can check that A is normal and $e_c(A)=3$ for every component C of D. After blowing up, on the chart U_1 we can consider the algebra $A_1=\pi^*A(U_1)\langle\, w_1,z_1\,\rangle$, where $w_1=v^{-1}y^2$, $z_1=v^{-1}xy$, and on the chart U_2 we can consider the algebra $A_2=\pi^*A(U_2)\langle\, w_2,z_2\,\rangle$, where $w_2=u^{-1}y^2$, $z_2=u^{-1}xy$. Again $A_1(U_1\cap U_2)=A_2(U_1\cap U_2)$, so we can glue them into a non-commutative surface $(\tilde{X},\tilde{A})$. One can verify that it is terminal. Let $\mathcal{I}$ be the locally bi-principal ideal in $\tilde{A}$ such that $\mathcal{I}(U_1)=(x)$ and $\mathcal{I}(U_2)=(x,w_2)$. Then $\tilde{A}/\mathcal{I}\simeq\mathcal{O}_Z$, where Z is the elliptic curve given by the equation $z_1^3=\xi(\xi-1)$ on U_1 and by $z_2^3=\eta(1-\eta)$ on U_2. Again x defines a global section of $\mathcal{I}$, hence of I, and $I\simeq\mathcal{O}_Z(\infty)$, where ∞ is the point on U_2 with $\eta=0$. Therefore, $\mathcal{I}$ is a good elliptic reduction cycle and Cohen–Macaulay modules over A are described in the same way as in Example 4.1. In particular, A is also Cohen–Macaulay tame.

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